

Critical exponents in Ising spin glasses

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We determine accurate values of ordering temperatures and critical exponents for Ising Spin Glass transitions in dimension 4, using a combination of finite size scaling and non-equilibrium scaling techniques. We find that the exponents η and z vary with the form of the interaction distribution, indicating non-universality at Ising spin glass transitions. These results confirm conclusions drawn from numerical data for dimension 3.

I. INTRODUCTION

Precise values of critical exponents at Ising Spin Glass (ISG) transitions have been hard to obtain from numerical simulations principally because of the agonizingly slow relaxation near the transition. Finite size scaling [1] can help bypass this problem but can introduce other difficulties, associated with the possible need to allow for corrections to the finite size scaling rules. We have shown [2] that in favourable cases a combination of finite size scaling and non-equilibrium scaling techniques can allow one to estimate to high accuracy both the ordering temperature and the exponents η and z . We exploited this method for ISGs in dimension 3 where we found that the exponents η and z changed considerably as a function of the distribution of near neighbour interactions, in complete contradiction to what would be expected from standard second order universality rules. There is no formal theorem which states that ISG transitions should obey universality rules, but up to now general sentiment tended to be in favour of universality. The empirical observation of non-universality [2] implies that the standard application of the renormalization group method to ISGs must be reconsidered, and shows that there are important qualitative differences between ISG transitions and conventional second order transitions.

We have now applied the same approach to ISGs in dimension 4 and for one case in dimension 5. We find that in dimension 4 as in dimension 3 the exponents η and z vary systematically from system to system. The numerical results for the exponents as functions of dimension can be compared with the predictions of the ϵ expansion; agreement is very poor if the expansion is extended beyond the leading term.

II. METHOD

Technically the problem of measuring exponents is less arduous in higher dimensions than in dimension 3. Because dimension 4 is well above the lower critical dimension the existence of a finite T_g in dimension 4 has never

been in doubt, in contrast to the situation in dimension 3. The Binder cumulant curves for each system show clear intersections and there already exist quite precise estimates of the ordering temperatures T_g , at least for the $\pm J$ and Gaussian interactions [3–7].

We will first outline the simulation techniques that we have used. First we measure, for a number of test temperatures T near the probable value of T_g , the time dependent non-equilibrium spin glass susceptibility $\chi^*(t)$. Two completely random (infinite temperature) replicas A and B of the same system are quenched to T with independent updating. The spin glass susceptibility

$$\chi^*(t) = [\langle S_i^A(t) S_i^B(t) \rangle^2] \quad (1)$$

is recorded as a function of time t after the start of the quench. Precisely at T_g , χ^* increases with t as t^h , with $h = (2 - \eta)/z$ [8]. If we do not know T_g a priori, we obtain an apparent $h(T)$ at each test temperature T . This non-equilibrium parameter presents the considerable advantage of requiring no preliminary anneal, and represents the growth of correlations as the internal temperature drops in the ISG samples. Because we concentrate on short times after having chosen an appropriately sized sample the correlation lengths are always much smaller than the sample size so there are no finite size corrections to a very good approximation. This type of non-equilibrium scaling has been tested very carefully on regular systems where the ordering temperature and the exponents were already known [9], and the scaling has been rigorously verified.

Secondly, at the same test temperatures T we anneal samples for a waiting time t_w before measuring the initial decay of the autocorrelation function over a further time t

$$q(t) = \langle S_i(t_w) S_i(t_w + t) \rangle \quad (2)$$

If $t \ll t_w$ and $T = T_g$, $q(t) = t^{-x}$ with $x = (d - 2 + \eta)/2z$ [10]. Under these conditions the decay of $q(t)$ is identical to the equilibrium form of the decay (which would be measured after an infinitely long anneal time), and again if the sample size is chosen appropriately there are no finite size corrections [11]. It turns out that that

at temperatures $T < T_g$, $q(t)$ follows an algebraic decay law with effective values of x which are smaller than the value corresponding to $T = T_g$. For higher temperatures there is a further multiplicative factor $f(t/\tau)$.

It can be seen that if reasonable precautions are taken the effective exponent combinations $h(T)$ and $x(T)$ can be measured with negligible systematic error. One can thus obtain a first set of effective exponents $\eta_1(T)$ and $z(T)$ as functions of the test temperature T from the equations

$$z(T) = \frac{d}{2x + h} \quad (3)$$

$$\eta_1(T) = \frac{4x - h(d - 2)}{2x + h} \quad (4)$$

Finally we use finite size scaling of the equilibrium spin glass susceptibility as a function of sample size L . Again at T_g the normalized spin glass susceptibility χ_{SG}/L^2 is proportional to $L^{-\eta}$ [1] giving an independent measure of η . Below T_g the power law form continues to hold, providing a second measure of η , $\eta_2(T)$. Now if we plot $\eta_1(T)$ and $\eta_2(T)$ against T , consistency dictates that the true T_g and η must correspond to the intersection point of $\eta_1(T)$ and $\eta_2(T)$.

III. RESULTS IN DIMENSION 4

We have made measurements for ISGs on the 4 dimension simple cubic lattice, with near neighbour interactions of different types. In practice the finite size scaling is the most demanding part of the simulation as far as CPU hours are concerned, as it is essential to make stringent checks that equilibrium has been achieved, and averages must be made over a large number of samples. We have been fortunate in being able to use equilibrium χ_{SG} data from A.P. Young [3] for the 4d $\pm J$ interaction case and from Parisi et al [5] for the 4d Gaussian interaction case. For the other two cases we carried out simulations in the usual manner for L up to 8 taking standard precautions [1] to be certain of complete thermal equilibrium at each size.

We thus dispose of data for the 4 dimension $\pm J$ (J), Uniform (U), Gaussian (G), and decreasing exponential (Ed) interactions. Earlier work [3–7] had shown that T_g is close to 2.0 for the $\pm J$ case and close to 1.75 for the Gaussian case. To get a trial estimates of T_g for the Ed and U cases we have relied on a Migdal-Kadanoff approach [12] where it is assumed that the normalization scaling parameter b is the same for all members of a family of ISGs in a given dimension. This gives quite reliable values of T_g once the value for one member of the family is known. Simulations for $h(T)$ were made using up to 2000 samples of size $L = 10$. $x(T)$ measurements were

made on samples of size $L = 20$ with anneals of 10^6 MCS and runs of 10^4 MCS. Runs were taken with one sample. Data points for $h(T)$ and $x(T)$ are shown in figure 1.

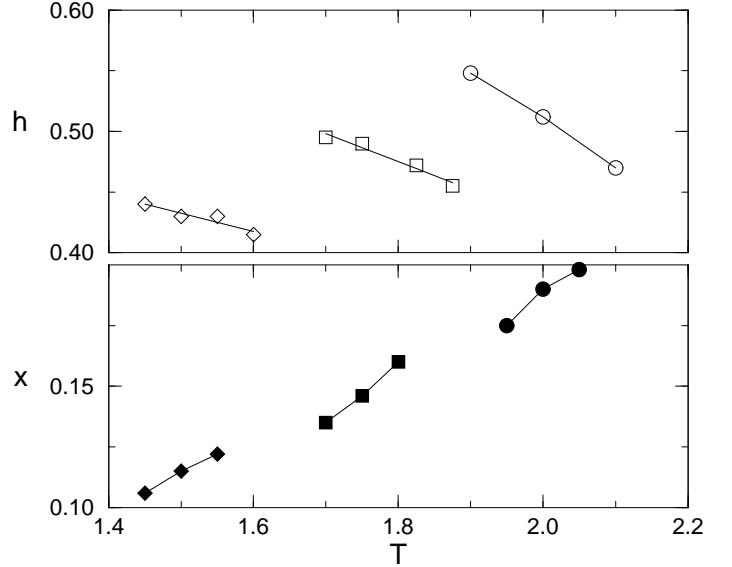


FIG. 1. The effective $h(T)$ and $x(T)$ (see text) as functions of temperature T for the $\pm J$ interaction (\circ), the Gaussian interaction (\square) and the decreasing exponential interactions (\diamond). Full symbols: x . Empty symbols: h .

It can be seen that for each system the effective value $x(T)$ increases with T while the effective value $h(T)$ decreases with T . In dimension 3, $h(T)$ increases with T [2]; we have no explanation for this difference.

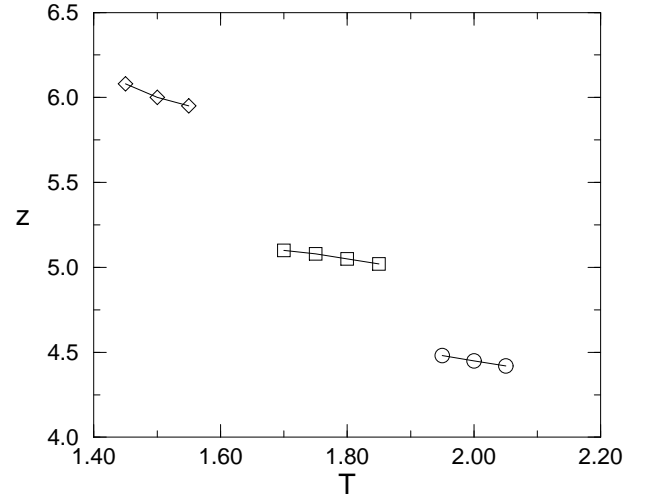


FIG. 2. The effective dynamic exponent z obtained using equation 3 for $\pm J$ interaction, gaussian interaction and decreasing exponential interactions. (same symbols as figure 1).

We first extract $z(T)$ from the $h(T)$ and $x(T)$ data using equation 3, figure 2. Clearly z is not universal; z tends to increase steadily from one type of system to the next as the kurtosis of the interaction distribution increases and T_g drops.

In figure 3 we have plotted $\eta_1(T)$ and $\eta_2(T)$ for the $\pm J$ case where we have the most complete η_2 data.

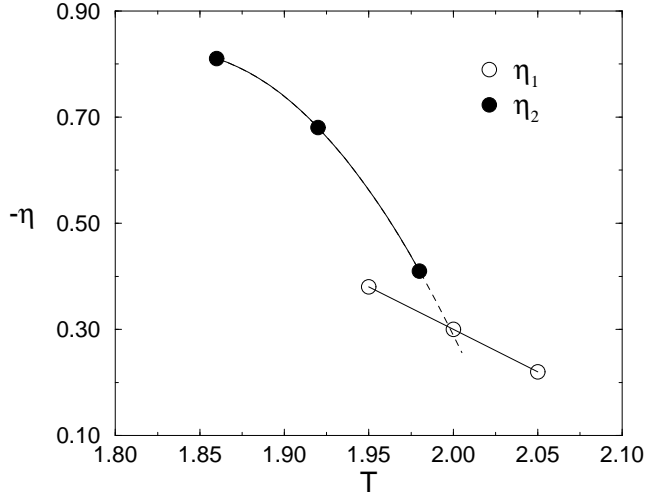


FIG. 3. Effective exponents $\eta_1(T)$ (\circ) and $\eta_2(T)$ (\bullet) (see text) as function of temperature for the $\pm J$ interaction case. The intersection gives T_g and the exponent η .

There is a clear intersection at the point $T_g = 2.00 \pm 0.01$ and $\eta = -0.30 \pm 0.02$. High quality Binder cumulant data for $L = 4$ to 12 due to Young [3] show a clearly defined crossing of all the $g_L(T)$ curves at $T = 2.00 \pm 0.01$ [14]. Two series expansion calculations gave T_g estimates of 2.02 ± 0.06 [6] and 2.04 ± 0.05 [7]. The agreement between different determinations of T_g in this case is thus very satisfactory, giving further confidence in the new method outlined above. We can note that for estimating η , this method is fairly insensitive to the exact value of T_g because the line for $\eta_1(T)$ is much less steep in the region of T_g than is the line for $\eta_2(T)$. Also the present method estimates η at the ordering temperature only, and so does not imply any assumption of a scaling relation over a range of temperatures around T_g .

If we now turn to the other sets of interaction distributions, Figure 4, we find similar intersections leading to the T_g and η values given in Table I.

The T_g value in the Gaussian interaction case lies between the two published estimates obtained from Binder cumulant work, and is close to estimates we can obtain starting from the $\pm J$ value for T_g and using the Migdal-Kadanoff technique [12] or by using the series expansion formula due to Singh and Fisher [15] (both methods give essentially $T_g = 1.75$ for the Gaussian case if T_g is taken

to be 2.00 for the $\pm J$ case).

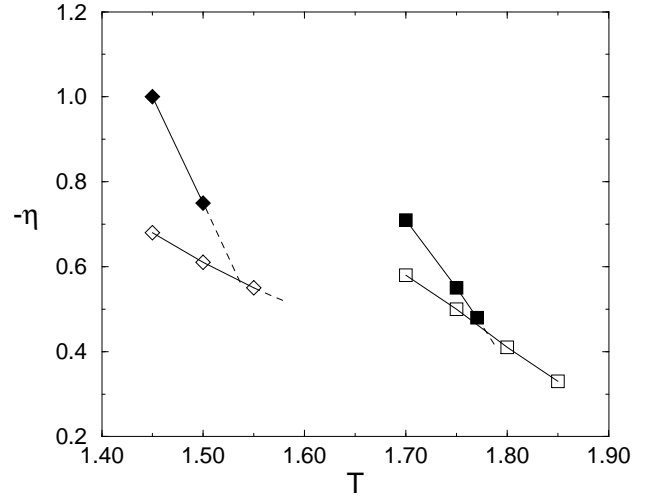


FIG. 4. As in figure 3, for the Gaussian interaction case (\square) and the decreasing exponential case (\diamond). Empty symbols: η_1 . Full symbol: η_2 .

For the U and Ed cases there are no Binder cumulant data to compare with but the Migdal-Kadanoff and Singh-Fisher techniques lead to T_g values of 1.90 and 1.53 respectively for the two interaction distributions. Overall agreement between the simulation estimates and these values can again be seen to be excellent. The η values, Table I, like the z values, are strikingly non-universal, with η becoming more negative as the kurtosis of the distribution increases and the T_g drops.

IV. DIMENSION 5

For the $\pm J$ interaction ISG in dimension 5, series expansion [7] gives a precise value for the ordering temperature, $T_g = 2.57 \pm 0.01$. We have seen above that in dimension 4 the simulations and the series work led to very similar T_g estimates, so the dimension 5 series estimate should be very reliable. We have assumed this T_g value is correct and have measured h and x at this T_g . The values, Table I, lead to $z = 4.50 \pm 0.1$ and $\eta = -0.39 \pm 0.02$, in excellent agreement with the series estimate $\eta = -0.38 \pm 0.07$.

We have also measured h for the Gaussian ISG in dimension 5 ($T_g = 2.31$), and for the $\pm J$ ISG in dimension 6 at T_g which is equal to 3.03 [7,16].

V. CRITICAL EXPONENTS - DIMENSIONAL DEPENDENCE.

We can first concentrate on the behaviour of the critical exponents and their combinations as a function of dimension for the series of ISGs with $\pm J$ interactions, in order to compare with ϵ expansion expressions [17,18]. In the standard renormalisation group treatment of the ISGs, the upper critical dimension is 6 and calculations have been made to third order in ϵ ($\epsilon = 6 - d$).

1/ $\eta(d)$. The values of η from [2] and the present work are shown in Figure 5, with an extrapolation to the upper critical dimension value $\eta = 0$ for dimension 6.

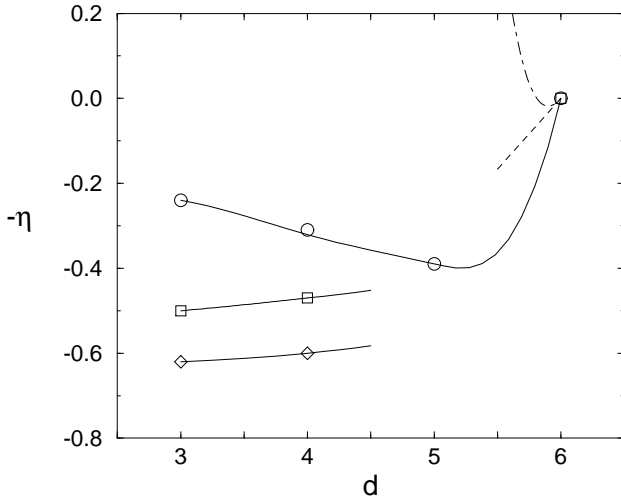


FIG. 5. η as a function of dimension for the $\pm J$ interaction (\circ), the Gaussian interaction (\square) and the decreasing exponential interactions (\diamond). The full lines are to guide the eye. The straight dashed line indicates the ϵ expansion to first order; the dot dashed curve is the ϵ expansion to order 3.

The renormalisation group ϵ expansion estimate to leading order in ϵ and to order 3 is also indicated. It can be seen that the simulation values, which from the discussion given above we consider very reliable, behave in a regular fashion. The initial trend of $\eta(d)$ extrapolated towards $d = 6$ is broadly consistent with the ϵ expansion to lowest order. However the ϵ expansion curve to order 3 lies a long way from the numerical points. This is in striking contrast to the standard second order transition case where the ϵ expansion to the same order gives excellent agreement with numerical or analytic values right down to $\epsilon = 2$.

2/ $h(d)$. Using van Hove arguments, Zippelius [18] found that that below the upper critical dimension $z = 2(2 - \eta)$, with no correction to leading order in ϵ . From the definition of h , this relation can be simply rewritten $h = 0.5$, so we would expect h to be close to this value

at and near to $d = 6$. Indeed we find that h is equal to 0.5 at $d = 6$, and that $h(d)$ does stay close to 0.5 down to dimension 4, Figure 6.

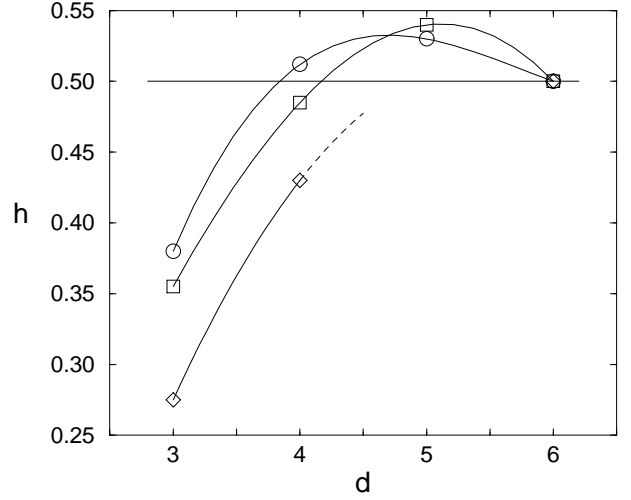


FIG. 6. The exponent h as a function of dimension for the $\pm J$ interaction (\circ), the Gaussian interaction (\square) and the decreasing exponential interactions (\diamond). The point at dimension 6 is measured.

This shows that the Zippelius relation between z and η is fairly accurate.

3/ $z(d)$. At dimension 6 we expect $z = 4$; the z data as a function of dimension are approaching 4 as dimension is increased towards 6, Figure 7.

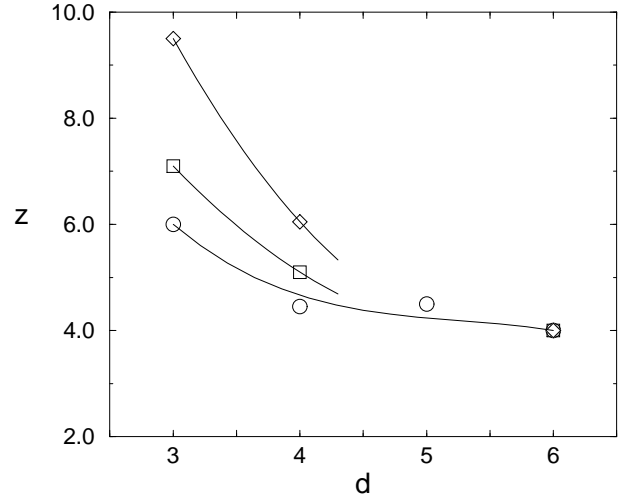


FIG. 7. The dynamic exponent z as a function of dimension for the $\pm J$ interaction (\circ), the Gaussian interaction (\square) and the decreasing exponential interactions (\diamond). The point at dimension 6 is the theoretical upper critical dimension value.

$4/\nu(d)$. We have no new information on ν from our simulations, but for completeness we present results compiled from various sources [3,7,10] in Figure 8 together with the ϵ expansion curve.

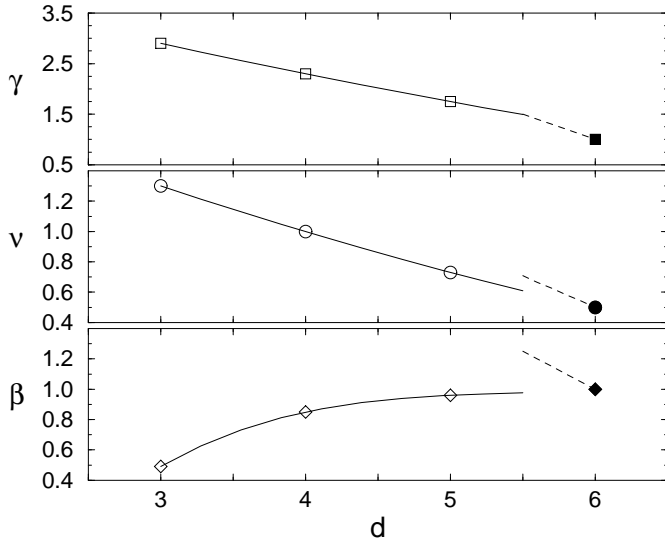


FIG. 8. A collection of critical exponents for the $\pm J$ interaction case as a function of dimension. The ν values are taken from references [3,7,10] and the γ and β values are calculated from η and ν using the standard scaling relations. The full points at $d = 6$ correspond to the theoretical upper critical dimension values and the dashed lines represent the theoretical ϵ expansions to first order.

Once η and ν are known, the other static exponents can all be deduced using the scaling rules, Figure 8.

VI. CRITICAL EXPONENTS - INTERACTION DEPENDENCE.

It can be seen in Table I that (as in dimension 3), the exponents in dimension 4 vary in a systematic manner, with η becoming more negative and z tending to a higher value when the kurtosis of the interaction distribution increases and the T_g drops. (There are many other possible parameters which could be used to modify the interaction distribution – for instance an alternative way to increase the kurtosis would be to dilute the interactions. We do not know if there is a one to one relationship between the exponents and the kurtosis in all cases).

We can attempt to understand from a naïve standpoint what the systematic behaviour is indicating [19]. If we make the heuristic assumption that the kurtosis is a pertinent parameter as far as the critical exponents are concerned, then as the kurtosis is increased and the T_g is driven towards zero, we would expect η and z to tend concomitantly towards their zero T_g values, which

are $2 - d$ and infinity respectively. If we plot, Figure 9, η as a function of T_g for dimensions 3 and 4, we find that the trend of the numerical values is consistent with a tendency for η towards $2 - d$ as T_g tends towards zero.

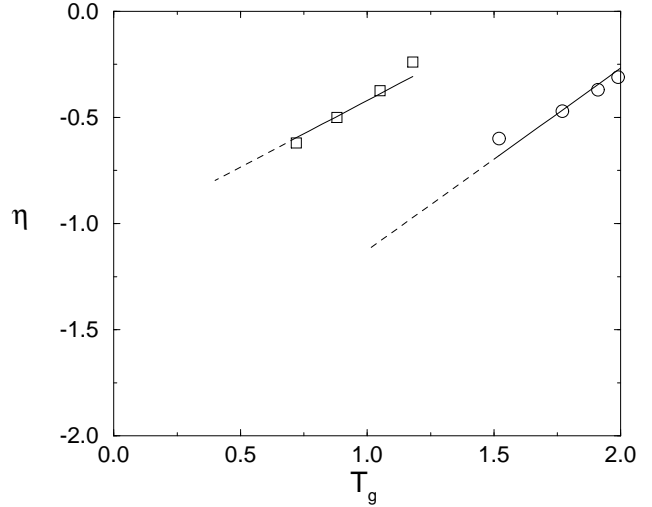


FIG. 9. The trend of η as a function of T_g for dimension 3 (circles) and dimension 4 (squares).

Plotting, Figure 10, $1/z$ against T_g we again find a trend corresponding to a divergence of z as T_g tends to zero.

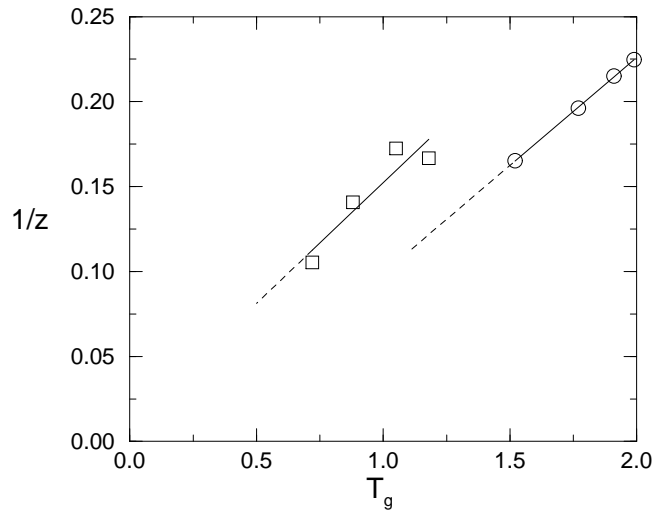


FIG. 10. $\frac{1}{z}$ plotted against T_g for dimension 3 (\square) and dimension 4 (\circ). $\frac{1}{z}$ extrapolates toward zero as T_g decreases, indicating that z diverges when T_g goes to zero.

We have so far studied the effect of just one parameter, the form of the interaction distribution, on the critical

exponents in finite dimension ISGs. Other parameters readily suggest themselves; it may well be that there is a continuous variation of the values of the critical exponents whenever any parameter is changed, so that the universality class concept simply does not apply to spin glasses. It is certainly relevant that the critical exponent z has been shown analytically to change continuously as a function of applied field along the AT line for the mean field spin glass [20]. It appears that the physics of transitions in complex systems such as spin glasses is fundamentally different from that of second order transitions in regular systems.

VII. CONCLUSION

We have found accurate values for the the ordering temperatures T_g and for the critical exponents η and z in Ising Spin Glasses with different sets of near neighbour interactions on the simple cubic lattice in dimension 4. The results demonstrate that, as in dimension 3 [2], the exponents are not universal, confirming that the universality class concept is not relevant to spin glass transitions. This observation shows that the spin glass transition cannot be treated as a second order transition in the conventional sense, and that the renormalization group theory as applied to spin glass transitions (and by extension to transitions in all complex systems) should be seriously reconsidered.

VIII. ACKNOWLEDGEMENTS

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d	System	T_g	$h(T_g)$	η	z
4	$\pm J$	1.99 ± 0.01	0.51 ± 0.02	-0.31 ± 0.01	4.45 ± 1
	U	1.91 ± 0.01	0.51 ± 0.02	-0.37 ± 0.02	4.65 ± 1
	G	1.77 ± 0.01	0.48 ± 0.02	-0.47 ± 0.02	5.10 ± 1
	Ed	1.52 ± 0.01	0.43 ± 0.02	-0.60 ± 0.03	6.05 ± 1
5	$\pm J$	2.57	0.530 ± 0.02	-0.39 ± 0.02	4.50 ± 1
	G	2.31	0.537 ± 0.02	—	—

TABLE I. Values of the temperature of transition and critical exponents for ISG for various distributions of interactions in dimension 4 and 5. The temperature of transition in dimension 5 where taken from series expansion

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